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Derivations of some important formulae in the theory of the fractional quantum Hall effect

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Abstract. Analytical derivations are given for the wave function of the quasi-hole excitation and for the relation between the filling factors of fermions and composite fermions. The extra terms which are added to the Hamiltonian of noninteracting electrons due to the presence of the composite-fermion vector potential are also investigated.

1. Introduction

In the last two decades, considerable effort has been devoted to establishing the theoretical foundation for the fractional quantum Hall effect (FQHE). In spite of this, the explanation of this strange phenomenon still needs further investigation and is subject to speculation. The present work is an attempt in this direction and its motivation is twofold. First, we introduce a new method for the derivation of the quasi-hole wave function. This wave function was originally introduced by Laughlin [1, 2]. It was chosen according to a physical argument which depends on the adiabatic addition of a flux quantum through an infinitely thin solenoid. In the present method, the fictitious-particles approach of Jain [3] is utilized and a Hilbert space for the wave functions of the particles in a certain species is defined. The Hilbert space used in the calculations resembles the one used in Girvin [4] for the wave function of the electrons in the lowest Landau level. The present approach has the advantage that it gives a general mathematical derivation for the wave function of a quasi-hole created at any point z_0 with a fractional charge e/m. This has not been performed before, as is clarified in section 2.

The second motivation for the work is to explore the explicit forms of the terms which arise due to the vector potential added in the theory of composite fermions (Jain [5]). The aim of such study is to predict the interaction that is responsible for the composite fermionization of the electrons, i.e. that causes the attachment of flux quanta to the electrons. The most important term was found to be of the form $1/r^2$ and not of the form $1/r$. The latter is the potential which should be expected due to Coulomb interactions between the electrons. We have further introduced an analogous vector potential for a single electron and presented a new approach that gave an adequate mathematical justification for Jain's relation between the filling factors of fermions and composite fermions.

The new derivation for the quasi-hole wave function is given in section 2. The investigation of the composite-fermion additional vector potential is considered in section 3.

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2. A new approach for creating quasi-holes

Laughlin [1, 2] suggested the following form of the quasi-hole wave function:

$$\Psi_m^{hz_0} = \prod_i^N (z_i - z_0) \prod_{j<k} (z_j - z_k)^m \exp\left(-\frac{1}{4}\sum_\ell |z_\ell|^2\right) \tag{1}$$

where N is the number of electrons, $z_j = x_j - \mathrm{i}y_j$ and z_0 is the position at which the quasi-hole has been created. He based his assumption on a physical point of view. Girvin [4] presented a method which gave a rigorous justification for this form when $m = 1$, i.e. in the simple case of the creation of a true hole. He further showed that the creation of m quasi-holes is locally equivalent to the creation of a true hole. Jain [3] used the fictitious-particles approach to derive the wave function of a quasi-hole created at the origin $z_0 = 0$ (the centre of a disc geometry). In the present section we combine the Girvin [4] and Jain [3] approaches to find a general proof for the above form of wave function. To the best of our knowledge this has not been done before. In the fictitious-particles approach, each electron is divided artificially into m distinct species (labelled by $\lambda = 1, 2, \ldots, m$). The Laughlin state is characterized by

$$\nu_\lambda = 1 \qquad e_\lambda = e/m \tag{2}$$

and

$$x_\lambda = \phi_1[z_j^{(\lambda)}] \exp\left[-\frac{1}{4}\frac{e_\lambda}{e}\sum_\ell |z_\ell^{(\lambda)}|^2\right] \qquad \lambda = 1, 2, \ldots, m \tag{3}$$

where

$$\phi_1[z_j^{(\lambda)}] = \prod_{j<k} (z_j^{(\lambda)} - z_k^{(\lambda)}). \tag{4}$$

ν_λ, e_λ and x_λ are the filling factor, the charge and the wave function of the fictitious particles of the species λ. Consequently, the electron wave function is given by the Laughlin wave function:

$$\psi_m = \prod_\lambda^m x_\lambda = \prod_{j<k} (z_j - z_k)^m \exp\left[-\frac{1}{4}\sum_\ell |z_\ell|^2\right]. \tag{5}$$

For a certain species $\tilde\lambda$, we define a Hilbert space analogous to the one defined in Girvin [4]. We thus include the exponential factor in equation (3) in the measure $\mathrm{d}\mu$ and define the Hilbert space of functions analytic in $z^{(\tilde\lambda)}$ via the inner product

$$\langle\theta, \phi\rangle = \int \mathrm{d}\mu(z^{(\tilde\lambda)})\, \theta^*(z^{(\tilde\lambda)})\phi(z^{(\tilde\lambda)}) \tag{6}$$

where

$$\mathrm{d}\mu(z^{(\tilde\lambda)}) = \frac{e_{\tilde\lambda}^2}{e^2}\frac{1}{2\pi\ell_0^2}\exp\left[-\frac{1}{2}\frac{e_{\tilde\lambda}}{e}|z^{(\tilde\lambda)}|^2\right]\mathrm{d}^2 z^{(\tilde\lambda)} = \frac{1}{2\pi\ell_{0\tilde\lambda}^2}\exp\left[-\frac{1}{2}|\tilde z^{(\tilde\lambda)}|^2\right]\mathrm{d}^2\tilde z^{(\tilde\lambda)} \tag{7}$$

$$\ell_{0\tilde\lambda} = \left(\frac{\hbar c}{e_{\tilde\lambda}B}\right)^{1/2} = \left(\frac{e}{e_{\tilde\lambda}}\right)^{1/2}\ell_0 \qquad \ell_0 = \left(\frac{\hbar c}{eB}\right)^{1/2} \tag{8}$$

and

$$\tilde z^{(\tilde\lambda)} = \left(\frac{e_{\tilde\lambda}}{e}\right)^{1/2} z^{(\tilde\lambda)} = \left(\frac{\ell_0}{\ell_{0\tilde\lambda}}\right) z^{(\tilde\lambda)}. \tag{9}$$

$z^{(\tilde\lambda)}$ measures the coordinates in units of the magnetic length ℓ_0 whereas $\tilde z^{(\tilde\lambda)}$ measures the coordinates in units of $\ell_{0\tilde\lambda}$. Also, the definition of the Hilbert space in equations (6)–(9) has been made for a general fictitious-particle state and not particularly a Laughlin state. For a Laughlin state we can further take $e_\lambda/e = 1/m$ as given in equation (2).

Now, we follow Girvin [4] and introduce the coherent state

$$\phi_{z_0}(z^{(\tilde\lambda)}) = \exp\left[\frac{e_{\tilde\lambda}}{2e} z_0^* z^{(\tilde\lambda)}\right] \tag{10}$$

which represents a Gaussian wave packet centred at the point z_0 (measured in units of ℓ_0). It also gives the projection on the lowest Landau level. A localized quasi-hole can thus be injected into a Laughlin state of fictitious particles by utilizing the coherent state (10). In this respect, we note from equations (3), (4) that the wave function representing $N+1$ fictitious particles of the species $\tilde\lambda$ in a Laughlin state is given by

$$\phi_1(z_1^{(\tilde\lambda)}, z_2^{(\tilde\lambda)}, \ldots, z_{N+1}^{(\tilde\lambda)}) = \prod_{j<k}^{N+1} (z_j^{(\tilde\lambda)} - z_k^{(\tilde\lambda)}) \tag{11}$$

after eliminating the exponential factor. Following a similar approach to that used in Girvin [4], the wave function which results after the injection of the quasi-hole is given by

$$\begin{aligned} \theta_{z_0}(z_1^{(\tilde\lambda)}, \ldots, z_N^{(\tilde\lambda)}) &= \langle \phi_{z_0}(z_{N+1}^{(\tilde\lambda)}), \phi_1(z_1^{(\tilde\lambda)}, \ldots, z_{N+1}^{(\tilde\lambda)})\rangle \\ &= \int \mathrm{d}\mu(z_{N+1}^{(\tilde\lambda)}) \exp\left[\frac{e_{\tilde\lambda}}{2e} z_0 z_{N+1}^{(\tilde\lambda)*}\right] \phi_1(z_1^{(\tilde\lambda)}, \ldots, z_{N+1}^{(\tilde\lambda)}). \end{aligned} \tag{12}$$

But, according to the Bargmann identity [6]:

$$\int \mathrm{d}\mu(z^{(\tilde\lambda)}) \exp\left[\frac{e_{\tilde\lambda}}{2e} z_0 z^{(\tilde\lambda)*}\right] \phi(z^{(\tilde\lambda)}) = \phi(z_0). \tag{13}$$

Consequently, it can be shown by applying (13) in (12) that

$$\theta_{z_0}(z_1^{(\tilde\lambda)}, \ldots, z_N^{(\tilde\lambda)}) = \phi_1(z_1^{(\tilde\lambda)}, \ldots, z_N^{(\tilde\lambda)}, z_0) = \prod_{i=1}^{N} (z_i^{(\tilde\lambda)} - z_0)\phi_1(z_1^{(\tilde\lambda)}, \ldots, z_N^{(\tilde\lambda)}). \tag{14}$$

Combining (14) with the wave functions of the other species (given by (3), (4)) and imposing the condition $z_j^{(\lambda)} = z_j$ (for all λ and j) we find

$$\psi_m^{hz_0}(z_1, \ldots, z_N) = \prod_{i=1}^{N} (z_i - z_0) \prod_{j<k} (z_j - z_k)^m \exp\left(-\frac{1}{4}\sum_\ell |z_\ell|^2\right) \tag{15}$$

which represents an electron state with a quasi-hole created at the point z_0. It is exactly identical to the form suggested by Laughlin (equation (1)).

3. The vector potential of composite fermions

In the composite-fermion approach (Jain [5]) the following vector potential:

$$\underline{\mathcal{A}}_j = 2m' \frac{\phi_0}{2\pi} \sum_{k, k\neq j} \underline{\nabla}_j \theta_{jk} \tag{16}$$

was added to the symmetric gauge vector potential

$$\underline{A}_j = \frac{1}{2} B(y_j \underline{e}_1 - x_j \underline{e}_2) \tag{17}$$

in order to attach to each electron $2m'$ flux quanta. In equation (17) $\underline{e}_1$ and $\underline{e}_2$ are unit vectors along the x- and y-axes while in equation (16) ϕ_0 is the unit of flux quanta hc/e and θ_{jk} is the angle subtended by the vector connecting the particles j and k with the x-axis. Accordingly

$$\theta_{jk} = \tan^{-1} \frac{y_j - y_k}{x_j - x_k} \tag{18}$$

and

$$\underset{\sim}{\mathcal{A}}_j = 2m' \frac{\phi_0}{2\pi} \sum_{k,k\neq j} \frac{1}{r_{jk}^2} [-(y_j - y_k)\underset{\sim}{e}_1 + (x_j - x_k)\underset{\sim}{e}_2]. \tag{19}$$

We now proceed to investigate the explicit forms of the terms which arise due to the additional vector potential $\underset{\sim}{\mathcal{A}}_j$. For such purposes we first note that the Hamiltonian operator now takes the form

$$\begin{aligned} H &= \frac{1}{2m_e} \sum_{j=1}^{N} \left(\frac{\hbar}{\mathrm{i}} \underset{\sim}{\nabla}_j - \frac{e}{c} \underset{\sim}{A}_j - \frac{e}{c} \underset{\sim}{\mathcal{A}}_j \right)^2 \\ &= H_0 + \frac{1}{2m_e} \sum_j \left(\frac{2e^2}{c^2} \underset{\sim}{A}_j \cdot \underset{\sim}{\mathcal{A}}_j + \frac{e^2}{c^2} \underset{\sim}{\mathcal{A}}_j \cdot \underset{\sim}{\mathcal{A}}_j + \frac{2\mathrm{i}\hbar e}{c} \underset{\sim}{\mathcal{A}}_j \cdot \underset{\sim}{\nabla}_j \right) \end{aligned} \tag{20}$$

where

$$H_0 = \frac{1}{2m_e} \sum_{j=1}^{N} \left(\frac{\hbar}{\mathrm{i}} \underset{\sim}{\nabla}_j - \frac{e}{c} \underset{\sim}{A}_j \right)^2$$

is the Hamiltonian of noninteracting electrons (before adding $\underset{\sim}{\mathcal{A}}_j$) and we have used the relation $\underset{\sim}{\nabla}_j \cdot \underset{\sim}{\mathcal{A}}_j = 0$ which follows from (19). It can further be shown after some algebra and by using (17), (19) that

$$\sum_j \underset{\sim}{A}_j \cdot \underset{\sim}{\mathcal{A}}_j = -\frac{m'\phi_0 B}{2\pi} \sum_j \sum_{k,k\neq j} \frac{1}{r_{jk}^2} [y_j(y_j - y_k) + x_j(x_j - x_k)] = -\frac{m'\phi_0 B}{4\pi} N(N-1) \tag{21}$$

$$\underset{\sim}{\mathcal{A}}_j \cdot \underset{\sim}{\mathcal{A}}_j = \frac{m'^2\phi_0^2}{\pi^2} \left\{ (N-1) \sum_{k\neq j} \frac{1}{r_{jk}^2} + \sum_{k\neq j} \sum_{k'\neq j} \frac{(x_j - x_{k'})(x_{k'} - x_k) + (y_j - y_{k'})(y_{k'} - y_k)}{r_{jk}^2 r_{jk'}^2} \right\} \tag{22}$$

and

$$\underset{\sim}{\mathcal{A}}_j \cdot \underset{\sim}{\nabla}_j = \frac{m'\phi_0}{\pi} \sum_{k,k\neq j} \frac{1}{r_{jk}^2} \left[-(y_j - y_k) \frac{\partial}{\partial x_j} + (x_j - x_k) \frac{\partial}{\partial y_j} \right]. \tag{23}$$

The terms in equations (21), (22), (23) do not involve any term which may give rise to a Coulomb interaction between the electrons. The only term which leads to an explicit interaction is the first term in (22). This term represents a repulsive potential of the form $1/r^2$. The repulsive nature of this term is consistent with the proposal usually made in the composite-fermion theory that attaching flux quanta to electrons can be driven by any convenient repulsive interaction. The problem of N particles undergoing $1/r^2$ interactions in the presence of an external magnetic field has been considered before in related contexts (Johnson and Quiroga [7]).

The coefficient of the $1/r^2$ term in equation (22) is given by

$$\alpha = \frac{1}{2m_e} \frac{e^2}{c^2} \frac{m'^2\phi_0^2}{\pi^2} (N-1) = \frac{2m'^2\hbar^2}{m_e} (N-1). \tag{24}$$

For a single electron attracted to a guiding centre at the origin (Asselmeyer and Keiper [8]) we may take

$$\alpha = \frac{2m'^2\hbar^2}{m_e} \tag{25}$$

in the thermodynamic limit. The Schrödinger equation of a single electron with an interaction potential α/r^2 can be solved exactly to find the corresponding eigenfunctions and eigenvalues.

To confirm further that the most important term which arises from $\underline{\mathcal{A}}_j$ is of the form $1/r^2$, we use the following alternative approach. In the single-electron calculations the form of the vector potential equivalent to $\underline{\mathcal{A}}_j$ may be taken as

$$\underline{\mathcal{A}} = 2m' \frac{\phi_0}{2\pi} \underline{\nabla}\phi \tag{26}$$

where ϕ is the polar angle of the electron position relative to the guiding centre. The single-electron Hamiltonian operators before (H_0) and after (H) adding $\underline{\mathcal{A}}$ are consequently given by

$$H_0 = \frac{-\hbar^2}{2m_e}\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right] - \frac{\mathrm{i}\hbar\omega_c}{2}\frac{\partial}{\partial\phi} + \frac{m_e\omega_c^2 r^2}{8} \tag{27}$$

and

$$H = H_0 - m'\hbar\omega_c + \mathrm{i}\frac{\hbar^2}{m_e}\frac{2m'}{r^2}\frac{\partial}{\partial\phi} + \frac{\hbar^2}{m_e}\frac{2m'^2}{r^2} \tag{28}$$

where ω_c is the cyclotron frequency eB/mc. The additional terms in (28) reveal that the vector potential $\underline{\mathcal{A}}$ still adds in the present approach an interaction potential of the form $1/r^2$.

Also, the addition of $\underline{\mathcal{A}}$ changes the wave function from ψ_0 to ψ, so

$$\psi = \psi_0 \exp(2\mathrm{i}m'\phi). \tag{29}$$

But, ψ_0 takes the form

$$\psi_0 = R_{nm}(r)\exp(\mathrm{i}m\phi) \tag{30}$$

and accordingly

$$\psi = R_{nm}(r)\exp(\mathrm{i}(m + 2m')\phi). \tag{31}$$

It is important to emphasize that m is the angular momentum quantum number (it determines the degeneracy of the state) while $2m'$ is the number of additional flux quanta attached to each electron. Furthermore, it is readily shown that

$$R_{nm}(r) = r^{|m|}\exp\left[-\frac{r^2}{4\ell_0^2}\right]U_n(r) \tag{32}$$

where $U_n(r)$ is a terminated series with highest power $2n$. The eigenvalue corresponding to both ψ_0 and ψ is consequently given by

$$E_{nm} = \hbar\omega_c\left[n + \frac{|m|+1}{2} + \frac{m}{2}\right] \tag{33}$$

where $m = -\gamma, \ldots, 0, \ldots$ for ψ_0 and $m = -(\gamma + 2m'), \ldots, 0, \ldots$ for ψ. Here γ is the degree of degeneracy before adding $\underline{\mathcal{A}}$. It thus follows that

$$\nu_0 = \frac{1}{\gamma} \qquad \nu = \frac{1}{\gamma + 2m'} = \frac{\nu_0}{1 + 2m'\nu_0} \tag{34}$$

where ν_0, ν are the filling factors before and after adding $\underline{\mathcal{A}}$. The final result in (34) is identical to Jain's relation between the filling factors of composite fermions and fermions [5]. The present approach thus leads to a simple justification for Jain's composite-fermion theory.

4. Conclusions

The method used in section 2 gave a reasonable analytical derivation for the quasi-hole wave function instead of the physical arguments used in earlier treatments. Moreover, it gives some support to the theory of fictitious particles. This theory was introduced by Jain in 1989 and was then forgotten. Also, the analysis made in section 3 showed that the additional vector potential considered in the composite-fermion approach does not give rise to a Coulomb interaction but it gives instead a potential of the form $1/r^2$. The analogous vector potential of a single electron introduced in this section has led to an adequate proof for the relation between the filling factors of fermions and composite fermions.

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